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NETWORKS MODELS AND ALGORITHMS FOR THREE TYPES OF VEHICLE ROUTING PROBLEMS

by

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ABSTRACT

This paper provides integer linear programming formulations for three types of vehicle routing problems. Appropriate relaxations of these formulations lend themselves to efficient treatment: the problems are solved by means of a network algorithm embedded in a branch and bound tree. Computational results are reported for two of these problems.

MODELES ET ALGORITHMES DE RESEAUX POUR TROIS TYPES DE PROBLEMES DE TOURNEES

RESUME

Cet article formule trois problèmes de tournée par la programmation linéaire en nombres entiers. On peut alors résoudre les problèmes de façon efficace à l'aide d'une relaxation appropriée, en imbriquant un algorithme de réseaux dans un arbre de type "branch and bound". On présente des résultats numériques pour deux de ces problèmes.

1. INTRODUCTION

This paper examines three extensions of the asymmetric travelling salesman problem (TSP) [2] and provides formulations and exact algorithms for their solution. In all three cases, the algorithms exploit the underlying network structure of these problems. We first define the TSP and the extensions considered.

Let $G = (N, A, C)$ be a directed graph where $N = \{1, \ldots, n\}$ is the set of nodes, A is the set of arcs and $C = (c_{ij})$, an asymmetric matrix of nonnegative distances associated with the arcs (i,j). C is said to satisfy the triangle inequality if and only if $c_{ij} \leq c_{ik} + c_{kj}$ for all $i,j,k \in N$. The TSP consists of determining the shortest circuit passing through each node of N once and only once. We also consider the following extensions of the TSP:

(STSP) Let $K = \{1, \ldots, \ell\}$ $(\ell \leq n)$, be a set of specified nodes. The *TSP with specified nodes* (STSP) consists of determining the shortest circuit passing through each node of K exactly once and through each node of N-K at most once. It is shown in [5] that the study of the STSP is only of interest if C does not satisfy the triangle inequality.

(CVRP) Let w_i be a non-negative weight associated with node i $(w_n = 0)$ and suppose the nodes are to be visited by m identical vehicles having the same capacity W. The *capacitated vehicle routing problem* (CVRP) [6,7] consists of establishing vehicle routes in such a way that (i) each vehicle strats and ends its journey at the same *depot* (node n, say), (ii) the sum of the weights on the route of any vehicle does not exceed W, (iii) each node of N-{n} is visited exactly once, (iv) the total distance travelled is minimized.

(GTSP) Now assume that N is partitioned into p clusters $T_1, \ldots, T_p$. The *generalized travelling salesman problem* (GTSP) [8] consists

of establishing a minimum distance circuit going through each cluster at least once and through each node at most once. It can be shown [8] that if C satisfies the triangle inequality, there exists an optimal solution containing only one node in each cluster. In [9], the authors demonstrate that the STSP constitutes a special case of the GTSP. Henry-Labordere [3] describes an application of the GTSP related to the optimal sequencing of record fields in a file for indexed access.

2. FORMULATIONS

Our aim is to devise exact algorithms for the STSP, the CVRP and the GTSP. If the problems are to be handled by integer linear programming, the formulation will inevitably contain a large number of constraints. Our approach is to initially relax most of these constraints in order to obtain a problem which can easily be handled by efficient network algorithms. The relaxed constraints can be gradually introduced into the problem (through a branch and bound scheme) and, as long as the network structure is preserved in each subproblem and relatively few of these subproblems have to be solved, the algorithms are likely to be fairly efficient. Such an approach has already been used by several authors for the TSP (see for example [2]) with a high degree of success. We propose a similar treatment for the asymmetric versions of the STSP, the CVRP and the GTSP. These three particular extensions were chosen as the treatment of the subproblems differs in each case. In all formulations, x_{ij} is a binary variable defined as follows:

(i) if $i \neq j$, x_{ij} indicates the presence ($x_{ij} = 1$) or the absence ($x_{ij} = 0$) of an arc from node i to node j in the optimal solution;

(ii) if $i = j$, x_{ij} indicates whether node i is used ($x_{ii} = 0$) or not ($x_{ii} = 1$) in the optimal solution.

In the TSP, the x_{ii}'s need not be defined. The usual formulation for this problem is:

$$\text{(TSP)} \qquad \text{minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\text{(D)} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad (j \in N)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i \in N)$$

$$\text{(S)} \quad \sum_{i,j \in S} x_{ij} \leq |S| - 1 \qquad (S \subset N,\ |S| \geq 2)$$

$$\text{(I)} \quad x_{ij} = 0 \text{ or } 1 \qquad (i,j \in N)$$

The following formulation for the STSP can be derived from Ibaraki's paper [4]. All the x_{ij}'s are defined and C is modified as follows: c_{ii} is set equal to 0 if $i \in N-K$ and to an arbitrarily large number M if $i \in K$. The formulation is similar to that of the TSP except that constraints (S) are replaced by

$$\text{(S)} \quad \sum_{i,j \in S} x_{ij} \leq |S| - 1 \qquad (S \subset N,\ S \cap K \neq K,\ S \cap K \neq \phi)$$

In the case of the CVRP, we first extend the distance matrix C into $C' = (c'_{ij})$ in the manner suggested by Lenstra and Rinnooy Kan [10]: node n is replaced by $\bar{m}$ nodes where $\bar{m}$ is an upper bound on m. Let $N' = N \cup \{n+1, \ldots, n+\bar{m}-1\}$. The new distances are defined as

$$c'_{ij} = \begin{cases} c_{ij} & (i,j \in N) \\ c_{in} & (j \in N' - N) \\ c_{nj} & (i \in N' - N) \\ 0 \text{ (for } m \leq \bar{m}) \text{ or } M \text{ (for } m = \bar{m}) & (i,j \in N' - N) \end{cases}$$

Also $w_i = 0$ for $i \in N'-N$. The CVRP formulation is obtained from that of the TSP by replacing n by $n+\bar{m}-1$ and N' by N; the right hand side of constraints (S) is also modified (see [6,7]):

$$(S) \qquad \sum_{i,j \in S} x_{ij} \leq |S| - \left\lceil \frac{\sum_{k \in S} w_k}{W} \right\rceil \qquad (S \subset N',\ |S| \geq 2)$$

where $\lceil t \rceil$ denotes the smallest integer greater than or equal to t if $t > 0$; $\lceil t \rceil = 1$ if $t \leq 0$.

Finally, the following formulation for the asymmetrical GTSP can be derived from [8]:

$$(GTSP) \qquad \text{minimize} \quad \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

subject to

$$(D) \qquad \sum_{i=1}^{n} x_{ik} = \sum_{j=1}^{n} x_{kj} = 1 \qquad (k \in N)$$

$$(S) \qquad \sum_{i \in T_k} x_{ii} \leq |T_k| - 1 \qquad (k = 1, \ldots, p)$$

(S) $\sum_{\substack{i,j\in S \\ i\neq j}} x_{ij} \leq |S| - 1$ ($S \subset N$, S has a non-empty intersection with at least 2 and at most $(n-1)T_k$'s)

(I) $x_{ij} = 0$ or 1 ($i,j \in N$)

All these formulations possess a similar structure: constraints (D) specify the degree of each node; constraints (S) prevent the formation of illegal subtours and ensure, in the case of the GTSP, that each T_k is visited at least once; constraints (I) are integrality constraints. Lack of space prevents us from explaining in detail the meaning of these constraints. Interested readers are referred to [5,9] for more extensive explanations.

3. ALGORITHMS

If in the above formulations, constraints (S) and (I) are relaxed, the resulting problem possesses a unimodular structure which can easily be handled by a network algorithm. In each case, the relaxed problem is in fact an assignment problem for which efficient algorithms exist [1].

The general method used to solve these problems can be summarized as follows:

Step 1: Obtain an upper bound on the value of the objective function by means of an appropriate heuristic. Solve the relaxed problem containing degree constraints only and upper bounds on the variables.

Step 2: Eliminate violated subtour elimination constraints by fixing the variables at 0 or 1 in a branch and bound tree. As in [2], branching is carried out on the illegal subtour containing the minimum number of arcs not already fixed at 1. In the STSP and in the GTSP, h subproblems are created by successively fixing at 0 the h variables not already fixed

at 1. In the CVRP, more than h subproblems can be generated since up to $r = \left\lceil \frac{\sum_{k \in S} w_k}{W} \right\rceil$ variables can be fixed at 0 simultaneously. (In fact, up to $\binom{h}{r}$ subproblems can be created.) In [6], the authors study various partitioning schemes for the asymmetrical CVRP. The process ends when all branches of the search tree have been explored.

It is worth noting that since the relaxed constraints (S) and (I) are satisfied by fixing the x_{ij}'s at 0 or at 1, the unimodular structure of the problem is preserved throughout the search tree. It can be seen that the STSP is a relaxation of the TSP since the occurence of illegal subtours is more limited. On the other hand, in the CVRP, more subproblems than in the TSP will have to be generated since illegal subtours are more likely to happen. This is especially true if W is small relative to $\sum_{k \in N} w_k$ (see [6,7]).

4. COMPUTATIONAL RESULTS

The above approach was tested on a number of randomly generated STSP's and CVRP's. Work on the GTSP is still in progress.

For the STSP, asymmetrical problems ranging from 80 to 200 nodes were obtained by randomly generating the c_{ij}'s on [0,100] according to a uniform distribution. Various values of ℓ were chosen. All problems were solved within 20 seconds on the University of Montreal Cyber 173 computer using an FTN4 compiler. Memory was the main limiting factor, some problems requiring more than the maximum memory allowed ($200\,000_8$ words).

For the CVRP, problems satisfying the triangle inequality and problems not possessing this property were generated. These were solved on the University of Montreal VAX 11/780 computer where extensive use was made of the virtual memory of that computer. Problems ranging

from 15 to 90 cities were solved successfully, taking for m the lowest feasible value, i.e. $L = \left\lceil \frac{\sum_{k \in N} w_k}{W} \right\rceil$. The largest problems solved contained 90 nodes. Better results were obtained by fixing m at L + 2 and L + 4: the largest problem attempted and solved successfully then contained 260 nodes and had a cost matrix which did not satisfy the triangle inequality. For these problems, our results indicate that much larger problems could have been solved. Computational times were highly variable and depended on various parameters of the problems, such as the relationship between W and $\sum_{k \in N} w_k$. The algorithms have succeeded in solving much larger STSP's and CVRP's than reported anywhere in the O.R. literature.

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